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ANALYTIC CONTINUATIONS OF GW THEORY

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ABSTRACT. This talk surveys my recent works with Yuan-Pin Lee and Hui-Wen Lin on analytic continuations of GW theory under simple ordinary flops [9, 6]. We first reduce the problem to local models via degeneration analysis. The genus zero theory (quantum cohomology) is then handled by classical mirror symmetry for toric varieties. The higher genus case follows from the genus zero result through sophesticated quantization procedure of semisimple theories. Explicit formulas of generating functions on extremal rays are given to demonstrate the result.

1. THE GROMOV–WITTEN THEORY

Let X be a complex projective manifold and $X_{g,n,\beta}$ the moduli space of stable maps $f : (C, p_1, \ldots, p_n) \to X$ with $g(C) = g$ and $[f(C)] = \beta \in NE(X)$. The virtual fundamental class $L_{g,n,\beta} \in A_D(X_{g,n,\beta}) \otimes \mathbb{Q}$ with expected dimension $D = (c_1(X).\beta) + (\dim X - 3)(1-g) + n$.

Let $0 \le m \le n$. If $m \ne 0$ it is required that $2g + m \ge 3$ (the stable range). The (n,m)-mixed invariants are

$$\Big\langle \prod_{i=1}^{m} \tau_{k_i,\bar{l}_i} a_i, \prod_{i=m+1}^{n} \tau_{k_i} a_i \Big\rangle_g = \int_{L_{g,n,\beta}} \prod_{i=1}^{m} \psi_i^{k_i} \bar{\psi}_i^{l_i} e_i^* a_i \prod_{i=m+1}^{n} \psi^{k_i} e_i^* a_i.$$

The insertion (field) at the i-th marked point comes from

(1) Primary fields: $a_i \in H(X)$ with $e_i : X_{g,n,\beta} \to X$ the evaluation map $e_i(f) = f(p_i)$.
(2) Descendants: $\psi_i = c_1(L_i)$ with L_i the universal cotangent line at the i-th section of the universal curve $X_{g,n+1,\beta} \to X_{g,n,\beta}$.
(3) Ancestors: $\bar{\psi}_i = \pi^* \psi_i$, with $\pi : X_{g,n,\beta} \to \overline{M}_{g,n} \to \overline{M}_{g,m}$ the stabilization followed by the forgetting map.

Let $\{T_i\}$ be a basis of $H(X, \mathbb{C})$, $g_{ij} = (T_i.T_j)$ and $\{T^i = \sum_j g^{ij} T_j\}$ the dual basis. The element $s \in H = H(X)$ is denoted by $s = \sum_i s^i T_i$, and the element $t \in \mathscr{H}_t = H[z] \cong \bigoplus_{k=0}^{\infty} H$ is denoted by $t = \sum_{k,i} t_k^i T_i \psi^k$. Then for any (n,m)-mixed insertion A we have generating (formal) functions

$$\langle A \rangle_g(t,s) = \sum_{\beta; n_1, n_2} \frac{q^\beta}{n_1! n_2!} \langle A, t^{\otimes n_1}, s^{\otimes n_2} \rangle_{g, n+n_1+n_2, \beta}.$$

Similarly this applies to $\bar{t} = \sum_{k,i} \bar{t}_k^i T_i \bar{\psi}^k \in \mathscr{H}_t$ in place of t. In particular for $A = \emptyset$ $(n = 0)$, we have descendant potentials

$$F_g^X(t) = \langle - \rangle_g(t); \qquad \mathscr{D}_X(t) = \exp \sum_{g=0}^{\infty} \hbar^{g-1} F_g^X(t).$$

The Gromov–Witten potential $F_g^X(s)$ is the restriction to $t = s \in H$. The quantum product uses genus 0 theory with $n \geq 3$ marked points. Namely

$$T_i *_s T_j = \sum_k \frac{\partial^3 F_0(s)}{\partial s^i \partial s^j \partial s^k}\, T^k = \sum_{n,\beta,k} \frac{q^\beta}{n!} \langle T_i, T_j, T_k, s^{\otimes n}\rangle_{0,3+n,\beta}\, T^k.$$

The associativity of the quantum product is equivalent to the WDVV equations–the flatness of the Dubrovin connection

$$\nabla_i^z = \frac{\partial}{\partial s^i} - \frac{1}{z} T_i *_s$$

on the (trivial) tangent bundle $TH \to H \ni s$, with $z \in \mathbb{C}^\times$ being a free parameter.

The GW potential is regraded as a function in two sets of variables. One is in the (complexified) Kähler moduli:

$$\omega = B + iH \in \mathscr{K}_X^{\mathbb{C}} = H^2(X,\mathbb{R})/H^2(X,\mathbb{Z}) + i\mathscr{K}_X,$$

with $\mathscr{K}_X$ being the Kähler cone, such that

$$q^\beta = e^{2\pi i \int_\beta \omega} = e^{2\pi i (B.\beta)} e^{-2\pi (H.\beta)}.$$

Conjecturally, $F_g(t)$ converges for H large. At ∞, $q^\beta = 0$ for $\beta \in NE(X)\backslash\{0\}$ and $*$ reduces to cup product. Alternatively one uses Novikov ring to work formally:

$$N(X) = \mathbb{C}[\widehat{NE(X)}],$$

the formal completion at the maximal ideal generated by all q^β with $\beta \neq 0$.

The second set of variables are s^i's. For similar convergence problem one views $\{s^i\}$ as formal variables and $F^g(s)$ as formal power series. This issue can be easily avoided since one usually work with n-pointed GW invariants instead of the full generating functions. The quantum cohomology ring $(QH(X), *)$ gives rise to a formal conformal Frobenius manifold $(H \otimes N(X), (.), *, \mathbf{1}, E)$ with $(,)$ the Poincaré pairing, $\mathbf{1}$ the fundamental class, and $E \in \Gamma(TH)$ the Euler vector field:

$$E = \sum_i (1 - \frac{1}{2}\deg T_i) s^i \frac{\partial}{\partial s^i} + c_1(T_X).$$

Similarly we have the ancestor potentials

$$\bar{F}_g^X(\bar{t}, s) = \langle - \rangle_g(\bar{t}, s); \qquad \mathscr{A}_X(\bar{t}, s) = \exp \sum_{g=0}^\infty \hbar^{g-1} \bar{F}_g^X(\bar{t}, s).$$

While descendants appear naturally in localization formula, the ancestors do have better functoriality. They are closely related:

Proposition 1.1. *In the stable range* $2g + n \geq 3$, *for* $(k_1, l_1) = (k+1, l)$,

$$\begin{aligned} &\langle \tau_{k+1,\bar{l}}\, a_1, \cdots \rangle_g(\bar{t}, s) \\ &\quad = \langle \tau_{k,\overline{l+1}}\, a_1, \cdots \rangle_g(\bar{t}, s) + \sum_\nu \langle \tau_k\, a_1, T_\nu \rangle_0(s)\, \langle \tau_{\bar{l}}\, T^\nu, \cdots \rangle_g(\bar{t}, s). \end{aligned} \tag{1.1}$$

2. Simple flops and quantum corrections

Let $\psi : X \to \bar{X}$ be a flopping contraction, with $\bar{\psi} : Z \cong P^r \to pt$ the restriction to the exceptional loci. Assume that $N_{Z/X} \cong \mathscr{O}_{P^r}(-1)^{\oplus(r+1)}$, then a simple P^r flop $f : X \dashrightarrow X'$ exists by blowing-up $\phi : Y = \mathrm{Bl}_Z X \to X$ followed by a blowing-down $\phi' : Y \to X'$ of the exceptoonal divisor $E \cong P^r \times P^r$ in the different direction:

$$\begin{tikzcd}
& E \subset Y \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] & \\
Z \subset X \arrow[rr, dotted, "f"] \arrow[dr, "\psi"'] & & Z' \subset X'. \arrow[dl, "\psi'"] \\
& pt \in \bar{X} &
\end{tikzcd}$$

The graph closure $[\bar{\Gamma}_f] = [Y] \in A^*(X \times X')$ induces a correspondence $\mathscr{F}$ which identifies $H(X)$ and $H(X')$ as well as the Poincare pairing. However the cup product is not preserved. Indeed let $H(Z) = H(P^r) = \mathbb{Z}[h]/(h^{r+1})$ and denote also by the same h a class in X which restricts to the hyperplane class of Z. For 3 classes $a_i \in A^{k_i}(X)$ with $1 \le k_i \le r$, $k_1 + k_2 + k_3 = \dim X = 2r + 1$, there is a topological defect of the cubic product:

$$(\mathscr{F}a_1.\mathscr{F}a_2.\mathscr{F}a_3)^{X'} - (a_1.a_2.a_3)^X = (-1)^r (a_1.h^{r-k_1})^X (a_2.h^{r-k_2})^X (a_3.h^{r-k_3})^X.$$

The quantum product comes in to rescue. It turns out that $QH(X)$ and $QH(X')$ are invariant under $\mathscr{F}$, after an analytic continuation in the extended Kähler moduli space. The analytic continuation is needed since $\mathscr{F}\mathscr{K}_X \cap \mathscr{K}_{X'} = \emptyset$. Equivalently for ℓ, ℓ' the extremal rays of ψ, ψ' respectively, $\mathscr{F}\ell = -\ell'$ and we identify

$$\mathscr{F}q^\beta = q^{\mathscr{F}\beta}; \qquad \mathscr{F}q^\ell = q^{-\ell'}$$

in the comparison of $\mathscr{F}F_0^X(s)$ and $F_0^{X'}(\mathscr{F}s)$. In doing so we have to first show that they are algebraic functions of q^ℓ and $q^{\ell'}$ respectively.

Let $a_i \in A^{k_i}(X)$ with $\sum k_i = 2r + 1 + (n-3)$. Using localization and classical mirror symmetry techniques (plus divisor reconstruction), we proved

Theorem 2.1 (Generalized multiple cover formula for $g = 0$)**.**

$$\langle a_1, \ldots, a_n \rangle_{n,d\ell}^X = (-1)^{(r+1)(d-1)} N_{k_1,\ldots,k_n} d^{n-3} (a_1.h^{r-k_1})^X \cdots (a_n.h^{r-k_n})^X.$$

Here $N_{k_1,\ldots,k_n}$'s are recursively defined universal constants.

To see this leads to quantum corrections to the cup product, consider the rational form of the geometric series

$$\mathbf{f}(q) := \frac{q}{1 - (-1)^{r+1} q} = \sum_{d \ge 1} (-1)^{(r+1)(d-1)} q^d.$$

It satisfies the functional equation $\mathbf{E}(q) := \mathbf{f}(q) + \mathbf{f}(q^{-1}) + (-1)^{r+1} = 0$ ($\mathbf{E}(q)$ is the formal Euler series), which will be the main source of analytic continuations.

Indeed, for extremal functions $\langle a \rangle_0 := \sum_{d=0}^\infty \langle a \rangle_{d\ell}\, q^{d\ell}$,

$$\begin{aligned}
&\langle \mathscr{F}a_1, \mathscr{F}a_2, \mathscr{F}a_3 \rangle_0^{X'} - \mathscr{F}\langle a_1, a_2, a_3 \rangle_0^X \\
&\quad = (a_1.h^{r-k_1})(a_2.h^{r-k_2})(a_3.h^{r-k_3})\big((-1)^r - \mathbf{f}(q^{\ell'}) - \mathbf{f}(q^{-\ell'})\big) = 0.
\end{aligned}$$

For $n \geq 4$, let $\delta = \delta_h = q^\ell \partial/\partial q^\ell = -q^{\ell'}\partial/\partial q^{\ell'} = -\delta_{h'}$ be the power operator. Then

$$\begin{aligned}&\langle \mathscr{F}a_1, \cdots, \mathscr{F}a_n\rangle_0^{X'} - \mathscr{F}\langle a_1, \cdots, a_n\rangle_0^X \\ &= (-1)^n N_{k_1,\ldots,k_n} \prod_{i=1}^n (a_i.h^{r-k_i})\, \delta^{n-3}\big(\mathbf{f}(q^{\ell'}) + \mathbf{f}(q^{-\ell'})\big) = 0.\end{aligned}$$

These explains the quantum corrections by the extremal rays.

Now we move to $g \geq 1$. The virtual dimension of $X_{g,n,d\ell}$ is given by

$$D_{g,n} = (\dim X - 3)(1-g) + n \tag{2.1}$$

which is independent of d. For $\alpha \in H(X)^{\otimes n}$, $d \geq 1$, $\langle \alpha\rangle_{g,n,d}$ depends only on the local geometry of $(Z, N_{Z/X})$. For $d = 0$ it depends on the global geometry of X.

If $g = 1$ then $D_{1,n} = n$. By the fundamental class axiom each cohomology insertion must be a divisor. Hence if $d \geq 1$ by the divisor axiom the n-point invariants are determined by $\langle - \rangle_{1,0,d}$. We omit the index n when no confusion may arise.

Let G be the genus one extremal function without marked points:

$$G(q) := \langle - \rangle_1 = \sum_{d\geq 0} \langle - \rangle_{1,d}\, q^d.$$

For $r = 1$ ($\dim X = 3$), $d \geq 1$, the formula $\langle - \rangle_{1,d} = 1/12d$ was first obtained by physical consideration in [1] and later mathematically justified in [5]. By using the theory of semisimple Frobenius manifolds, we generalize it to all $r \in \mathbb{N}$:

Theorem 2.2 (Generalized multiple cover formula for $g = 1$)**.** *For $d \in \mathbb{N}$,*

$$\langle - \rangle_{1,d} = (-1)^{d(r+1)} \frac{r+1}{24d}. \tag{2.2}$$

Equivalently,

$$\delta_h G = \sum_{d\geq 1} \langle h \rangle_{1,d\ell}\, q^d = (-1)^{r+1}\frac{r+1}{24}\mathbf{f}. \tag{2.3}$$

We will verify that $\mathscr{F}\langle\alpha\rangle_1^X = \langle \mathscr{F}\alpha\rangle_1^{X'}$ by showing that the genus one invariants with $d \geq 1$ correct the semi-classical defect from $d = 0$: $\langle\alpha\rangle_{1,0}^X - \langle\mathscr{F}\alpha\rangle_{1,0}^{X'}$.

If $n \geq 2$, the divisor axiom shows that $\langle\alpha\rangle_{1,n,0} = 0$ and in fact there is no defect. Now $\delta_{h'} = -\delta_h$ and $\delta_h \mathbf{f}(q) = \delta_{h'}\mathbf{f}(q')$, hence $\delta_h^2 G(q) = \delta_{h'}^2 G'(q')$ and then

$$\langle h, \ldots, h\rangle_{1,n}^X = \delta_h^n G(q) = (-1)^{n-2}\delta_{h'}^n G'(q') = (-1)^n \langle h', \ldots, h'\rangle_{1,n}^{X'}.$$

Since $\mathscr{F}h^k = (-1)^k h'^k$, this implies the invariance for all $n \geq 2$.

If $n = 1$, it is well known that $X_{g,n,0} \cong X \times \overline{M}_{g,n}$ and

$$L_{g,n,0} = e(\mathscr{E}) \cap [X \times \overline{M}_{g,n}], \tag{2.4}$$

where the obstruction bundle is given by $\mathscr{E} = \pi_1^* T_X \otimes \pi_2^* \mathcal{H}_g^\vee$ with $\mathcal{H}_g$ being the Hodge bundle. Let $\lambda_i = c_i(\mathcal{H}_g)$. For $(g, n) = (1, 1)$ we have

$$e(\mathscr{E}) = c_{\rm top}(X) - c_{{\rm top}-1}(X).\lambda_1.$$

Thus for one point invariant we get a semi-classical term

$$\langle a\rangle_{1,0}^X = -(c_{{\rm top}-1}(X).a)^X \cdot \int_{\overline{M}_{1,1}} \lambda_1 = -\frac{1}{24}(c_{{\rm top}-1}(X).a)^X, \tag{2.5}$$

where the basic Hodge integral $\int_{\overline{M}_{1,1}} \lambda_1 = 1/24$ is used.

For $n = 1$, to prove the invariance we may assume that X and X' are projective local models $P_{P^r}(\mathscr{O}(-1)^{r+1} \oplus \mathscr{O})$. We compute via (2.5), (2.3) that

$$\langle h\rangle_1^X - \langle \mathscr{F}h\rangle_1^{X'} = -\frac{1}{24}\left((c_{2r}(X).h) - (c_{2r}(X').(\xi' - h'))\right) - \frac{r+1}{24}.$$

Since $X \cong X'$, the invariance follows from the topological calculation:

$$(c_{2r}(X).(2h - \xi)) = -(r+1).$$

For $g \geq 2$, we must have $\dim X = 3$ and then $D_{g,n} = n$. As in the $g = 1$ case we are reduced to consider the case $n = 0$. For simple P^1 flop, the extremal invariants with $d \geq 1$ are determined by Faber and Pandharipande [2] to be

$$\langle -\rangle_{g,d} = C_g\, d^{2g-3}$$

where $C_g = |\chi(M_g)|/(2g-3)!$. The generating function

$$\langle -\rangle_g := \sum_{d=0}^{\infty} \langle -\rangle_{g,d}\, q^d = \langle -\rangle_{g,0} + C_g\, \delta^{2g-3}\mathbf{f}, \tag{2.6}$$

is invariant under $\mathscr{F}$ since $2g - 3 \geq 1$ and $\langle -\rangle_{g,0}^X = \langle -\rangle_{g,0}^{X'}$: The local models of X and X' are both isomorphic to $P_{P^1}(\mathscr{O}(-1)^2 \oplus \mathscr{O})$, and hence have the same degree zero invariants.

With the above motivations, now we state the main result:

Theorem 2.3. *For a simple flop f, any generating function of mixed invariants of f-special type*

$$\langle \tau_{k_1,\bar{l}_1}\alpha_1, \cdots, \tau_{k_n,\bar{l}_n}\alpha_n\rangle_g,$$

with $2g + n \geq 3$, is invariant under $\mathscr{F}$ up to analytic continuation under the identification of Novikov variables $\mathscr{F}q^\beta = q^{\mathscr{F}\beta}$. In particular,

$$\mathscr{F}\mathscr{A}_X(\bar{t}, s) \cong \mathscr{A}_{X'}(\mathscr{F}\bar{t}, \mathscr{F}s).$$

Here a mixed insertion $\tau_{k_j,\bar{l}_j}\alpha_j$ consists of descendents ψ_j^k and ancestors $\bar{\psi}_j^l$. Given $f : X \dashrightarrow X'$ with exceptional loci $Z \subset X$ and $Z' \subset X'$, a mixed invariant is of f-special type if for every insertion $\tau_{k_j,\bar{l}_j}\alpha_j$ with $k_j \geq 1$ we have $\alpha_j.Z = 0$.

3. Degeneration reduction to local models

Given a pair (Y, E) with $E \hookrightarrow Y$ a smooth divisor, let $\Gamma = (g, n, \beta, \rho, \mu)$ with $\mu = (\mu_i) \in \mathbb{N}^\rho$ a partition of $(\beta.E) = |\mu| := \sum_{i=1}^{\rho} \mu_i$. For $A \in H^*(Y)^{\oplus n}$, $k, l \in \mathbb{Z}_+^n$ and $\varepsilon \in H^*(E)^{\oplus\rho}$, we require that $2g + n + \rho \geq 3$ if $l \neq 0$, and then the *mixed relative invariant* of stable maps with topological type Γ (i.e. with contact order μ_i in E at the i-th contact point) is given by

$$\langle \tau_{k,\bar{l}}A \mid \varepsilon, \mu\rangle_\Gamma^{(Y,E)} = \int_{[\overline{M}_\Gamma(Y,E)]^{virt}} \Big(\prod_{j=1}^n \psi_j^{k_j}\bar{\psi}_j^{l_j} e_{Y,j}^* A^j\Big) \cup e_E^*\varepsilon,$$

where $e_{Y,j} : \overline{M}_\Gamma(Y, E) \to Y$, $e_E : \overline{M}_\Gamma(Y, E) \to E^\rho$ are evaluation maps on marked points and contact points respectively.

If $\Gamma = \coprod_\pi \Gamma^\pi$, the relative invariants with possibly disconnected domain curves are defined by the product rule:

$$\langle \tau_{k,\bar{l}}A \mid \varepsilon, \mu\rangle_\Gamma^{\bullet(Y,E)} = \textstyle\prod_\pi \langle (\tau_{k,\bar{l}}A)^\pi \mid \varepsilon^\pi, \mu^\pi\rangle_{\Gamma^\pi}^{(Y,E)}.$$

It is set to be zero if some ancestor in the right hand side is *undefined*. This is the case when there is a π with $l_{\Gamma^\pi} \neq 0$ but $g_{\Gamma^\pi} = 0$, $n_{\Gamma^\pi} = \rho_{\Gamma^\pi} = 1$.

Consider a degeneration $W \to \mathbb{A}^1$ of a trivial family with $W_t \cong X$ for $t \neq 0$ and $W_0 = Y_1 \cup Y_2$ a simple normal crossing. All classes $\alpha \in H^*(X, \mathbb{Z})^{\oplus n}$ have global lifting and for each lifting the restriction $\alpha(0)$ on W_0 is defined. Let $j_i : Y_i \hookrightarrow W_0$ be the inclusion maps. The lifting is encoded by (α_1, α_2) with $\alpha_i = j_i^* \alpha(0)$.

Let $\{\varepsilon_i\}$ be a basis of $H^*(E)$ with $\{\varepsilon^i\}$ its dual basis. $\{\varepsilon_I\}$ forms a basis of $H^*(E^\rho)$ with dual basis $\{\varepsilon^I\}$ where $|I| = \rho$, $\varepsilon_I = \varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_\rho}$.

The *degeneration formula* expresses the absolute invariants of X in terms of the relative invariants of the two smooth pairs (Y_1, E) and (Y_2, E):

Theorem 3.1 ([10])**.** *Assume that $2g + n \geq 3$ if $l \neq 0$, then*

$$\langle \tau_{k,\bar{l}} \alpha \rangle^X_{g,n,\beta} = \sum_I \sum_{\eta \in \Omega_\beta} C_\eta \langle \tau^1_{k,\bar{l}} \alpha_1 \mid \varepsilon_I, \mu \rangle^{\bullet (Y_1, E)}_{\Gamma_1} \langle \tau^2_{k,\bar{l}} \alpha_2 \mid \varepsilon^I, \mu \rangle^{\bullet (Y_2, E)}_{\Gamma_2}. \tag{3.1}$$

Here $\eta = (\Gamma_1, \Gamma_2, I_\rho)$ is an *admissible triple* which consists of (possibly disconnected) topological types $\Gamma_i = \coprod_{\pi=1}^{|\Gamma_i|} \Gamma_i^\pi$ with the same partition μ of contact order under the identification I_ρ of contact points. The constants $C_\eta = m(\mu)/|\mathrm{Aut}\,\eta|$, where $m(\mu) = \prod \mu_i$ and $\mathrm{Aut}\,\eta = \{\, \sigma \in S_\rho \mid \eta^\sigma = \eta \,\}$. Ω_β is the set of equivalence classes of all admissible triples with fixed degree β.

The first step is to apply deformation to the normal cone

$$W = \mathrm{Bl}_{Z \times \{0\}} X \times \mathbb{A}^1 \to \mathbb{A}^1.$$

$W_0 = Y_1 \cup Y_2$, $Y_1 = Y = \mathrm{Bl}_Z X \xrightarrow{\phi} X$ and $Y_2 = \tilde{E} = P_Z(N_{Z/X} \oplus \mathscr{O}) \xrightarrow{p} Z$. $E = Y \cap \tilde{E}$ is the ϕ exceptional divisor as well as the infinity divisor of $\tilde{E}$.

Similar construction applies to X':

$$W' = \mathrm{Bl}_{Z' \times \{0\}} X' \times \mathbb{A}^1 \to \mathbb{A}^1.$$

$W'_0 = Y'_1 \cup Y'_2$, $Y'_1 = Y' = \mathrm{Bl}_{Z'} X' \xrightarrow{\phi'} X'$, $Y'_2 = \tilde{E}' = P_{Z'}(N_{Z'/X'} \oplus \mathscr{O}) \xrightarrow{p'} Z'$ and $E' = Y' \cap \tilde{E}'$. By the construction of P^r flops we have $(Y, E) = (Y', E')$. For simple P^r flops we even have $\tilde{E} \cong \tilde{E}'$ as both are $P_{P^r}(\mathscr{O}(-1)^{\oplus(r+1)} \oplus \mathscr{O})$. However $W_0 \ncong W'_0$ since the gluing of $\tilde{E}$ to Y along $E \cong P^r \times P^r$ differs from the one of $\tilde{E}'$, with the P^r factors switched.

In fact, the flop f induces $f_{loc} : X_{loc} = \tilde{E} \dashrightarrow X'_{loc} = \tilde{E}'$, the projective local model of f, which is again a simple P^r flop.

Define the generating series for genus g (connected) relative invariants

$$\langle A \mid \varepsilon, \mu \rangle^{(\tilde{E}, E)}_g := \sum_{\beta_2 \in NE(\tilde{E})} \frac{1}{|\mathrm{Aut}\,\mu|} \langle A \mid \varepsilon, \mu \rangle^{(\tilde{E}, E)}_{g, \beta_2} q^{\beta_2}. \tag{3.2}$$

Proposition 3.2 (Reduction to relative local models)**.** *To prove*

$$\mathscr{F} \langle \tau_{k,\bar{l}} \alpha \rangle^X_g = \langle \tau_{k,\bar{l}} \mathscr{F} \alpha \rangle^{X'}_g$$

for all $\alpha \in H^*(X)^{\oplus n}$ and $k, l \in \mathbb{Z}^n_+$, it suffices to show

$$\mathscr{F} \langle \tau_{k,\bar{l}} A \mid \varepsilon, \mu \rangle^{(\tilde{E}, E)}_{g_0} = \langle \tau_{k,\bar{l}} \mathscr{F} A \mid \varepsilon, \mu \rangle^{(\tilde{E}', E)}_{g_0}$$

for all $A \in H^*(\tilde{E})^{\oplus n}$, $k, l \in \mathbb{Z}^n_+$, $\varepsilon \in H^*(E)^{\oplus \rho}$, contact type μ, and all $g_0 \leq g$.

This follows form the degeneration formula by matching the data on the (Y, E) side. Then we have further reductions to local absolute invariants:

Proposition 3.3 ([9], Proposition 4.8). *For the local simple flop $\tilde{E} \dashrightarrow \tilde{E}'$, to prove*

$$\mathscr{F}\langle \tau_{\bar{l}} A \mid \varepsilon, \mu\rangle_g^{(\tilde{E},E)} = \langle \tau_{\bar{l}} \mathscr{F} A \mid \varepsilon, \mu\rangle_g^{(\tilde{E}',E)}$$

for all $A \in H^*(\tilde{E})^{\oplus n}$, $l \in \mathbb{Z}_+^n$, and weighted partitions (ε, μ), it suffices to show for mixed invariants of special type

$$\mathscr{F}\langle \tau_{\bar{l}} A, \tau_k \varepsilon\rangle_{g_0}^{\tilde{E}} = \langle \tau_{\bar{l}} \mathscr{F} A, \tau_k \varepsilon\rangle_{g_0}^{\tilde{E}'}$$

for all A, l, ε and $k \in \mathbb{Z}_+^\rho$, and all $g_0 \leq g$.

The idea of proof is by induction on $(g, |\mu|, n, \rho)$ with ρ in the reverse ordering. We degenerate a suitable absolute invariant of f-special type (with virtual dimension matches) so that the desired relative invariant appears as the main term:

$$\begin{aligned}\langle \tau_{\bar{l}} A, \tau_{\mu_1 - 1}\varepsilon_{i_1}, \ldots, \tau_{\mu_\rho - 1}\varepsilon_{i_\rho}\rangle_g^{\bullet\tilde{E}} &= \sum_{\mu'} m(\mu') \times \\ \sum_{I'} \langle \tau_{\mu_1 - 1}\varepsilon_{i_1}, \ldots, \tau_{\mu_\rho - 1}\varepsilon_{i_\rho} \mid \varepsilon^{I'}, \mu'\rangle_0^{\bullet(Y_1,E)} &\langle \tau_{\bar{l}} A \mid \varepsilon_{I'}, \mu'\rangle_g^{(\tilde{E},E)} + R,\end{aligned}$$

where R denotes the remaining terms, and to show this is an invertible system.

4. Semisimple Frobenius manifolds and quantization

Let $N = \dim H(X)$. The quantum cohomology differential equation

$$\nabla^z S = 0 \tag{4.1}$$

has a fundamental solution at $z = \infty$: $S = (S_{\mu,\nu}(s, z^{-1}))$, an $N \times N$ matrix-valued function, in (formal) power series of z^{-1} satisfying the conditions

$$S(s, z^{-1}) = Id + O(z^{-1}) \quad \text{and} \quad S^*(s, -z^{-1})S(s, z^{-1}) = Id, \tag{4.2}$$

where * denotes the adjoint with respect to $(\cdot, \cdot)$. S is essentially the *big J function*.

How about at $z = 0$? A point $s \in H$ is called a *semisimple* point if the quantum product on the tangent algebra $(T_s H, *_s)$ at $s \in H$ is isomorphic to $\bigoplus_1^N \mathbb{C}$ as an algebra. $(QH, *)$ is called semisimple if the semisimple points are (Zariski) dense in H. If $(QH, *)$ is semisimple, it has idempotents $\{\epsilon_i\}_1^N$

$$\epsilon_i * \epsilon_j = \delta_{ij}\epsilon_i.$$

defined up to S_N permutations. The *canonical coordinates* $\{u^i\}_1^N$ is a local coordinate system on H near s defined by $\partial/\partial u^i = \epsilon_i$. When the Euler field is present, the canonical coordinates are also uniquely defined up to signs and permutations. We will often use the *normalized* form $\tilde{\epsilon}_i = \epsilon_i/\sqrt{(\epsilon_i, \epsilon_i)}$. It is easy to see that $\{\epsilon_i\}$ and $\{\tilde{\epsilon}_i\}$ form orthogonal bases: $(\epsilon_i, \epsilon_j) = (\epsilon_i * \epsilon_i, \epsilon_j) = (\epsilon_i, \epsilon_i * \epsilon_j) = \delta_{ij}(\epsilon_i, \epsilon_i)$.

When the quantum cohomology is semisimple, (4.1) has a fundamental solution at $z = 0$ of the following type

$$\mathbf{R}(s, z) := \Psi(s)^{-1} R(s, z)\, e^{\mathbf{u}/z},$$

where $(\Psi_{\mu i}) := (T_\mu, \tilde{\epsilon}_i)$ is the transition matrix from $\{\tilde{\epsilon}_i\}$ to $\{T_\mu\}$; $\mathbf{u}$ is the diagonal matrix $(\mathbf{u}_{ij}) = \delta_{ij}u^i$. The main information of $\mathbf{R}$ is carried by $R(s, z)$, which is a (formal) power series in z. See [4] and Theorem 1 in Chapter 1 of [8].

Let $\mathscr{H}_{\mathbf{q}} := H[z]$, $\{T_\mu z^k\}_{k=0}^\infty$ be a basis of $\mathscr{H}_{\mathbf{q}}$, and $\{\mathbf{q}_k^\mu\}$ the dual coordinates. We define an affine isomorphism of $\mathscr{H}_{\mathbf{q}}$ to $\mathscr{H}_t$ via a *dilaton shift* "$t = \mathbf{q} + z\mathbf{1}$":

$$t_k^\mu = \mathbf{q}_k^\mu + \delta^{\mu 1}\delta_{k1}. \tag{4.3}$$

The cotangent bundle $\mathscr{H} := T^*\mathscr{H}_{\mathbf{q}}$ is naturally isomorphic to the H-valued Laurent series in z^{-1}, $H[z][\![z^{-1}]\!]$. It has a natural symplectic structure

$$\Omega = \sum_{k,\mu,\nu} g_{\mu\nu}\, d\mathbf{p}_k^\mu \wedge d\mathbf{q}_k^\nu$$

where $\{\mathbf{p}_k^\mu\}$ are the dual coordinates in the fiber direction of $\mathscr{H}$ in the natural basis $\{T_\mu(-z)^{-k-1}\}_{k=0}^\infty$. In this way, $\Omega(f,g) = \mathrm{Res}_{z=0}(f(-z), g(z))$.

To quantize an infinitesimal symplectic transformation on $(\mathscr{H}, \Omega)$, or its corresponding quadratic hamiltonians, we recall the standard Weyl quantization. An identification $\mathscr{H} = T^*\mathscr{H}_{\mathbf{q}}$ of the symplectic vector space $\mathscr{H}$ as a cotangent bundle of $\mathscr{H}_{\mathbf{q}}$ is called a polarization. The "Fock space" will be a certain class of functions $f(\hbar, \mathbf{q})$ on $\mathscr{H}_{\mathbf{q}}$ (containing at least polynomial functions), with additional formal variable $\hbar$ ("Planck's constant"). The classical observables are certain functions of $\mathbf{p}, \mathbf{q}$. The quantization process is to find for the phase space of the "classical mechanical system" on $(\mathscr{H}, \Omega)$ a "quantum system" on the Fock space such that the classical observables, like the hamiltonians $h(\mathbf{q}, \mathbf{p})$ on $\mathscr{H}$, are quantized to become operators $\widehat{h}(\mathbf{q}, \partial/\partial\mathbf{q})$ on the Fock space.

Let $A(z)$ be an $\mathrm{End}(H)$-valued Laurent formal series in z satisfying

$$\Omega(Af, g) + \Omega(f, Ag) = 0,$$

for all $f, g \in \mathscr{H}$. That is, $A(z)$ defines an infinitesimal symplectic transformation. $A(z)$ corresponds to a quadratic "polynomial" hamiltonian $P(A)$ in $\mathbf{p}, \mathbf{q}$

$$P(A)(f) := \frac{1}{2}\Omega(Af, f).$$

Choose a *Darboux coordinate system* $\{\mathbf{q}_k^i, \mathbf{p}_k^i\}$ so that $\Omega = \sum d\mathbf{p}_k^i \wedge d\mathbf{q}_k^i$. The quantization $P \mapsto \widehat{P}$ assigns

$$\widehat{1} = 1, \quad \widehat{\mathbf{p}_k^i} = \sqrt{\hbar}\frac{\partial}{\partial \mathbf{q}_k^i}, \quad \widehat{\mathbf{q}_k^i} = \mathbf{q}_k^i/\sqrt{\hbar} \tag{4.4}$$

which extends multiplicatively to quadratic polynomials. In summary, the quantization is the process

$$\begin{array}{ccccc} A & \mapsto & P(A) & \mapsto & \widehat{P(A)} \\ \text{inf. sympl. transf.} & \mapsto & \text{quadr. hamilt.} & \mapsto & \text{operator on Fock sp..} \end{array}$$

The first map is a Lie algebra isomorphism: The Lie bracket on the left is defined by $[A_1, A_2] = A_1A_2 - A_2A_1$ and the Lie bracket in the middle is the Poisson bracket. The second map is close to be a Lie algebra homomorphism.

For example, let $\dim H = 1$ and $A(z) = z^{-1}\times$. Then $A(z)$ is infinitesimally symplectic and

$$P(z^{-1}) = -\frac{\mathbf{q}_0^2}{2} - \sum_{m=0}^\infty \mathbf{q}_{m+1}\mathbf{p}_m; \qquad \widehat{P(z^{-1})} = -\frac{\mathbf{q}_0^2}{2} - \sum_{m=0}^\infty \mathbf{q}_{m+1}\frac{\partial}{\partial \mathbf{q}_m}. \tag{4.5}$$

One often has to quantize symplectic transformations. We define $\widehat{e^{A(z)}} := e^{\widehat{A(z)}}$ for $A(z)$ an infinitesimal symplectic transformation.

Let $\mathscr{D}_N(\mathbf{t}) = \prod_{i=1}^N \mathscr{D}_{pt}(t^i)$ be the descendent potential of N points, where

$$\mathscr{D}_{pt}(t^i) \equiv \mathscr{A}_{pt}(t^i) := \exp \sum_{g=0}^{\infty} \hbar^{g-1} F_g^{pt}(t^i)$$

is the total descendent potential of a point and $t^i = \sum_k t_k^i z^k$.

Suppose that $(QH, *)$ is semisimple and $N = \dim H$. Since $\{\tilde{e}_i\}$ defines an *orthonormal basis* for $T_s H \cong H$ (for s a semisimple point), the dual coordinates $(\mathbf{p}_k^i, \mathbf{q}_k^i)$ of the basis $\{\tilde{e}_i z^k\}_{k\in\mathbb{Z}}$ for $\mathscr{H}$ form a Darboux coordinate system. The coordinate system $\{t_k^i\}$ is related to $\{\mathbf{q}_k^i\}$ by the dilaton shift (4.3). Note that $\partial/\partial \mathbf{q}_k^i = \partial/\partial t_k^i$. The following beautiful formula was formulated by Givental [4] and recently established by C. Teleman in a preprint [11].

Theorem 4.1. *For X a smooth variety with semisimple $(QH(X), *)$,*

$$\mathscr{A}_X(\bar{t}, s) = e^{\bar{c}(s)} \Psi^{-1}(s) \widehat{R}_X(s, z) \widehat{e^{\mathbf{u}/z}}(s) \mathscr{D}_N(\mathbf{t}), \tag{4.6}$$

where $\bar{c}(s) = \frac{1}{48} \log \det(\epsilon_i, \epsilon_j)$. The term in $\mathbf{u}$ can be removed by the string equation.

In particular, for semisimple quantum cohomology the higher genus theory is determined by the genus zero theory and the case of points.

5. Semi-Fano/Semi-Simple toric local model

To proceed to the idea of proof of Theorem 2.3, it is crucial to think about analytic continuations, or $\mathscr{F}$-invariance, in a more algebraic manner. Let NE_f be the $\mathscr{F}$-effective cone of classes $\beta \in NE(X)$ with $\mathscr{F}\beta \in NE(X')$. Define the ring

$$\mathscr{R} = \widehat{\mathbb{C}[NE_f]}[\mathbf{f}],$$

which can be regarded as certain algebraization of $N(X)$ in the q^ℓ variable. $\mathscr{R}$ is canonically identified with its counterpart $\mathscr{R}' = \widehat{\mathbb{C}[NE'_f]}[\mathbf{f}']$ under $\mathscr{F}$ since $\mathscr{F} NE_f = NE'_f$ and $\mathscr{F}\mathbf{f}(q^\ell) = (-1)^r - \mathbf{f}(q^{\ell'})$. In general, the analytic continuation is understood as an identity in $\mathscr{R} \cong \mathscr{R}'$.

We start with $g = 0$ without ancestors. Since $\tilde{E} = P_{P^r}(\mathscr{O}(-1)^{r+1} \oplus \mathscr{O})$, $H^*(\tilde{E}) = \mathbb{Z}[h, \xi]/\langle h^{r+1}, (\xi - h)^{r+1}\xi\rangle$ is generated by divisors where $\xi = [E]$. $NE(\tilde{E}) = \mathbb{Z}\ell \oplus \mathbb{Z}\gamma$ with γ being the fiber line of $\tilde{E} \to Z$. Write $\beta = d\ell + d_2\gamma$. By a dimension count, for each $\alpha \in \tau_\bullet H^*(\tilde{E})^{\oplus n}$, $\langle \alpha \rangle_\beta^{\tilde{E}} \neq 0$ for at most one d_2. Then the process in [9] via the *reconstruction theorem* and induction on $d_2 \geq 0$ shows that there are indeed only two relations which generate all the analytic continuations.

The first is $\mathbf{E}(q) = 0$, which is the origin of analytic continuation: For $d_2 = 0$, this is essentially the quantum corrections discussed in §2. Another relation comes from the *quasi-linearity* $\mathscr{F}\langle \tau_k \xi a\rangle^X = \langle \tau_k \xi' \mathscr{F} a\rangle^{X'}$ for one point f-special invariants. This is an identity of *small J functions* in $\mathbb{C}[NE'_f]$: $\mathscr{F} J^{\tilde{E}}.\xi a = J^{\tilde{E}'}.\xi' \mathscr{F} a$, where no analytic continuation is needed by the semi-Fano mirror theorem .

The proof also shows that the analytic continuations arise from finite $\mathbb{C}[NE_f]$-linear combinations of $\delta^m \mathbf{f}'$s with $m \geq 0$. Moreover, $\delta^m \mathbf{f}$ is a polynomial in $\mathbf{f}$. This follows from $\delta\mathbf{f} = \mathbf{f} + (-1)^{r+1}\mathbf{f}^2$ and $\delta(\mathbf{f}_1\mathbf{f}_2) = (\delta\mathbf{f}_1)\mathbf{f}_2 + \mathbf{f}_1\delta\mathbf{f}_2$. Hence all the corresponding generating functions in q^ℓ and $q^{\ell'}$ are identified in $\mathscr{R}$.

Now we consider $g \geq 0$ with ancestors. Denote by $q_1 = q^\ell$ and $q_2 = q^\gamma$, then

$$QH^*_{small}(X) \cong \mathbb{C}[h,\xi][q_1,q_2]/(h^{r+1} - q_1(\xi - h)^{r+1}, (\xi - h)^{r+1}\xi - q_2).$$

Solving the relations, we get the eigenvalues of $h*$ and $\xi*$ are

$$h = \eta^j\omega^i q_1^{\frac{1}{r+1}} q_2^{\frac{1}{r+2}} (1 + \omega^i q_1^{\frac{1}{r+1}})^{-\frac{1}{r+2}}, \quad \xi = \eta^j q_2^{\frac{1}{r+2}} (1 + \omega^i q_1^{\frac{1}{r+1}})^{\frac{r+1}{r+2}} \tag{5.1}$$

for $i = 0, 1, \cdots, r$ and $j = 0, 1, \cdots, r+1$, where ω and η are the $(r+1)$-th and the $(r+2)$-th root of unity respectively. As these eigenvalues are all different, we see that $h*$ and $\xi*$ are semisimple operators, hence $QH^*_{small}(X)$ is semisimple.

This proves that $(QH^*, *)$ is semisimple at the origin $s = 0$. Since semisimplicity is an open condition, the formal Frobenius manifold $QH^*(X)$ is also semisimple.

Now we apply the quantization formula (Theorem 4.1) over the ring $\mathscr{R}$ and conclude the $\mathscr{F}$ invariance of pure ancestor invariants. To handle mixed invariants of f-special type, we apply Proposition 1.1 and reduce the problem to one-point descendant of f-special type with $g = 0$, a case which has already been solved.

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