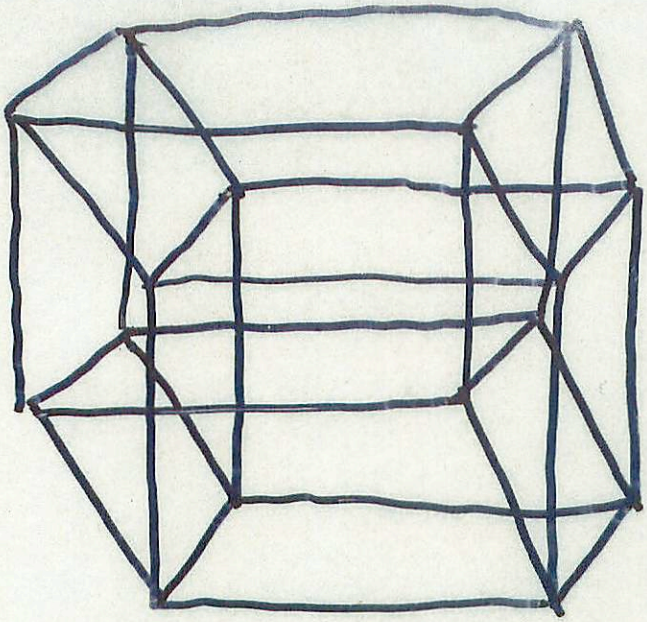


$$\{\text{HEAD, TAIL}\}^N$$

=

LENGTH - N SEQUENCES

=

N - DIM CUBE

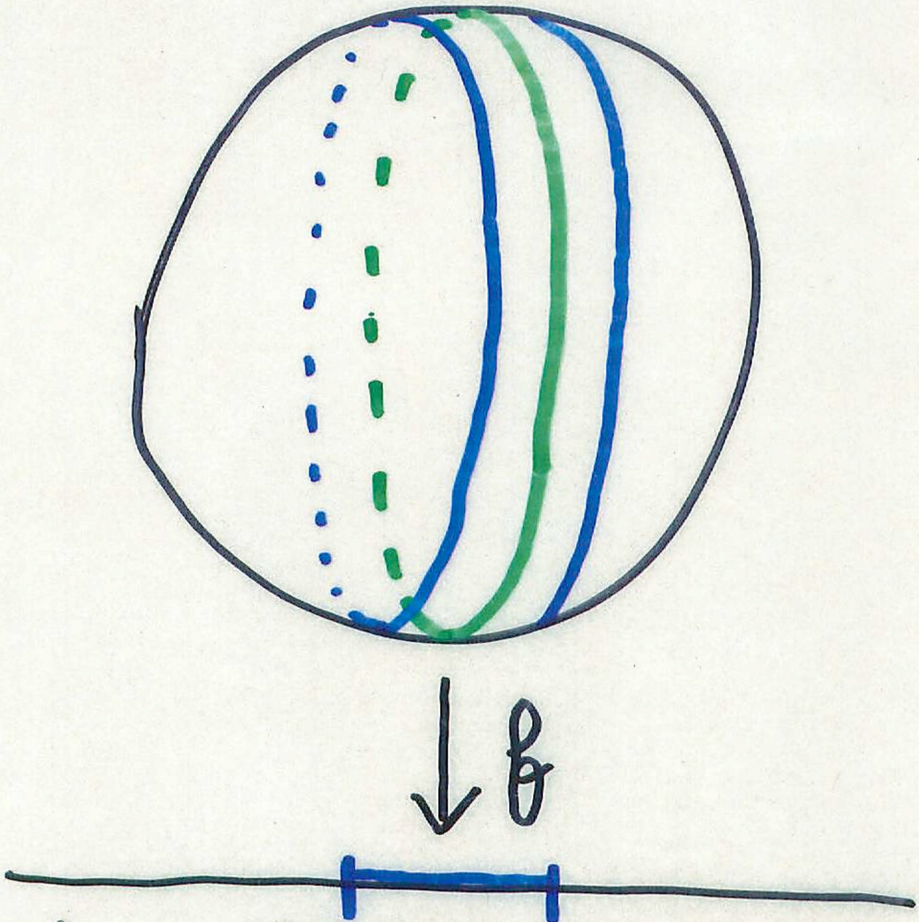
$$f := \frac{1}{N} \ \#\ \text{"HEADS" IN THE SEQUENCE}$$

THEN

$$f = \frac{1}{2} \pm \frac{1}{\sqrt{N}}$$

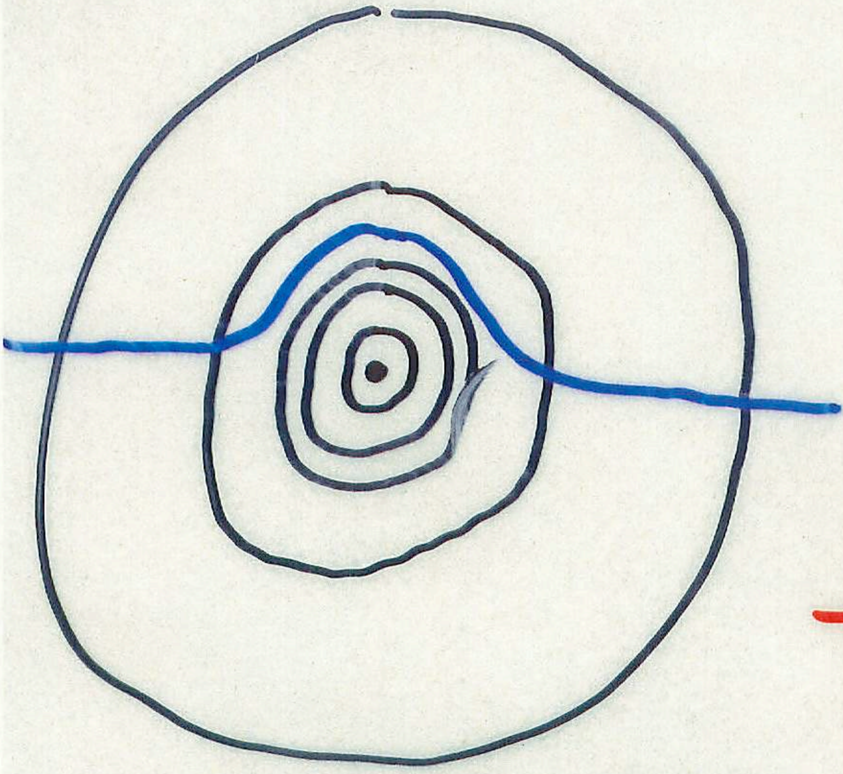
$S^{N-1} \subset \mathbb{R}^N$

UNIT SPHERE



$f =$ PROJECTION ON AN AXIS

THEN $f = 0 \pm \dfrac{1}{\sqrt{N}}$



$\mathbb{R}^N$ WITH GAUSSIAN MEASURE

$$\frac{e^{-|x|^2/2V^2}}{(\sqrt{2\pi}\,V)^N}$$

V SUCH THAT $\mathbb{E}|x|^2 = 1$

X AN N-DIM. RIEM. MANIFOLD WITH RICCI $\geqslant$ RICCI (S^N)

MORE GENERALLY, $\forall f$ 1-LIPSCHITZ $\exists m \in \mathbb{R}$, $f = m \pm \frac{1}{\sqrt{N}}$

MORE PRECISELY,

$$\mu\left(x,\ |f(x) - m| > tD\right) \leq e^{-\frac{t^2}{2}}$$

WITH $D \asymp 1/\sqrt{N}$

SIGN OF CURVATURE

→ **NEGATIVE CURVATURE**

≈ HYPERBOLIC PLANE $\mathbb{H}^N$

NEGATIVE **SECTIONAL** CURVATURE

DISCRETE VERSIONS:

δ - HYPERBOLIC SPACES,

CAT(0), CAT(-1), ...

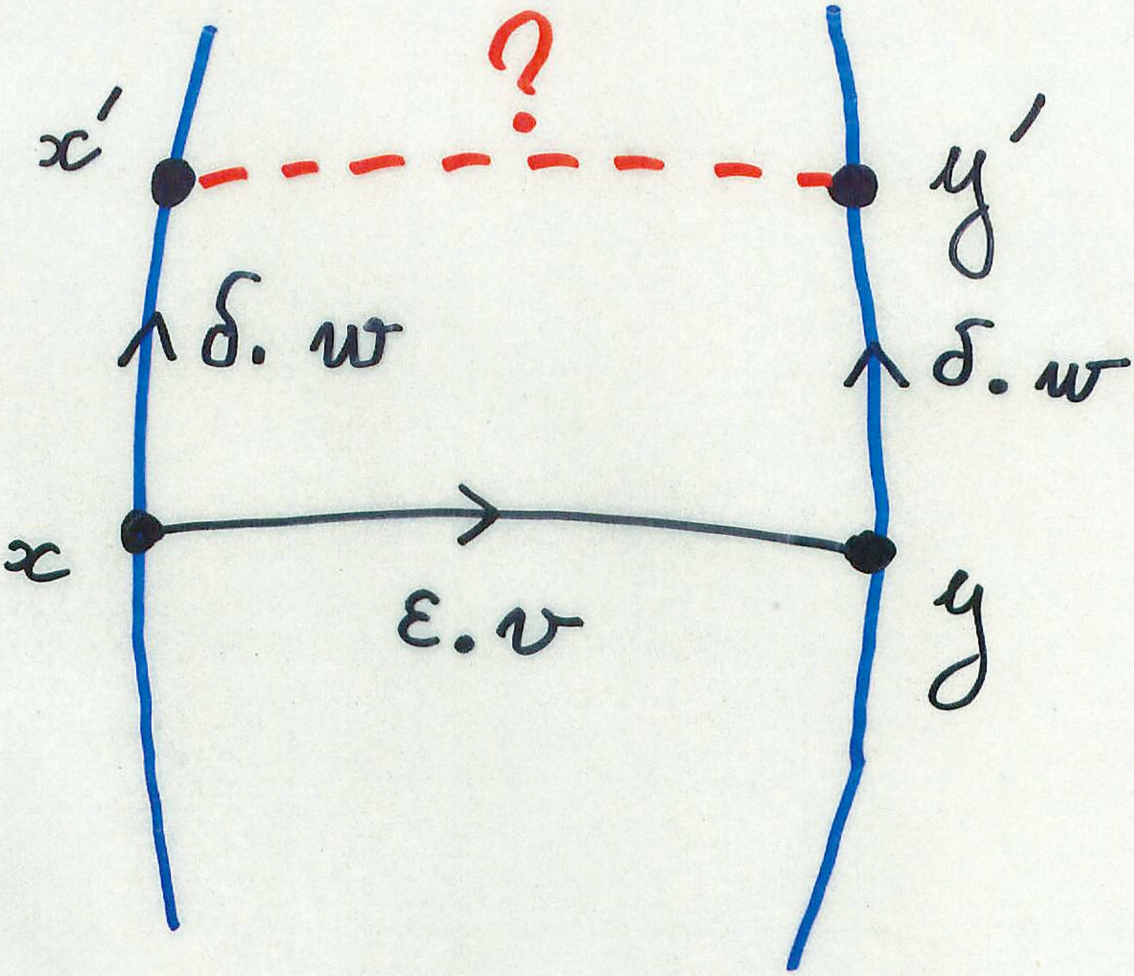
→ **POSITIVE CURVATURE**

≈ SPHERE S^N

POSITIVE **RICCI** CURVATURE

DISCRETE VERSION: ?

SECTIONAL CURVATURE



$$d(x', y') = d(x, y) \times \left(1 - \frac{\delta^2}{2} K(v, w)\right)$$

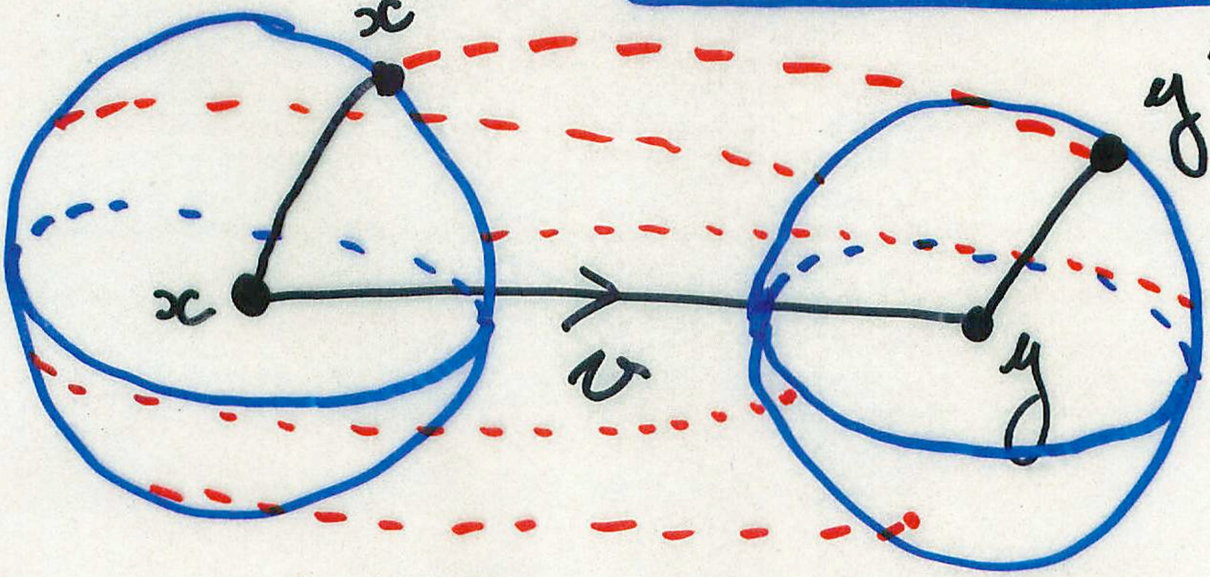
$K(v, w) =$ **SECTIONAL CURVATURE**

IN PLANE v, w

RICCI CURVATURE

$$\mathrm{Ric}(v) := N \int_{w \,\in\, \text{TANGENT UNIT SPHERE}} K(v, w)$$

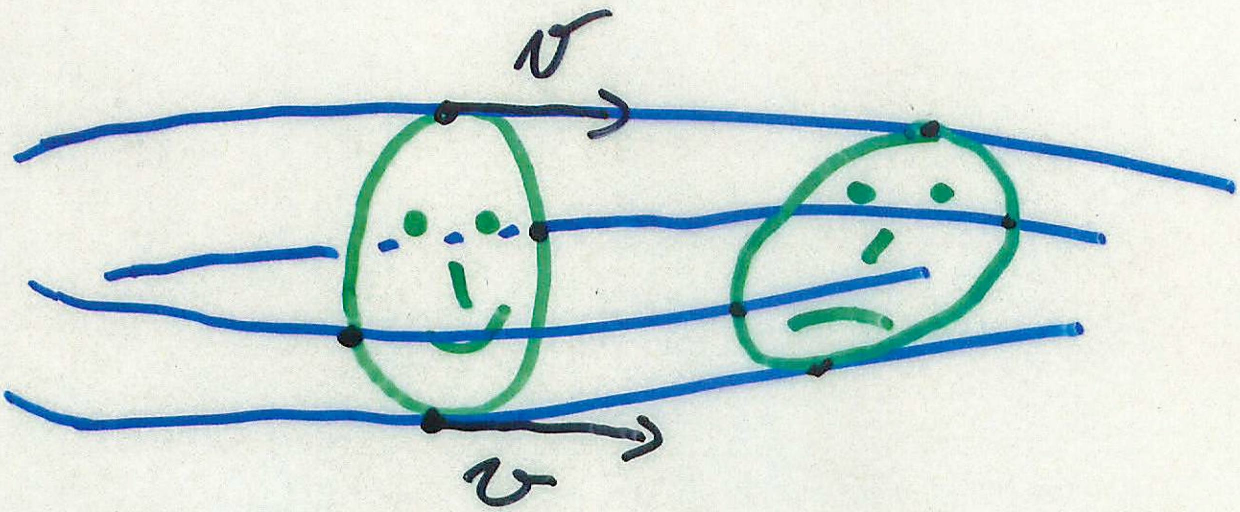
"BALLS ARE CLOSER THAN THEIR CENTERS ARE"
$\Longleftrightarrow$ RIC > 0



$$d(x', y') = d(x, y) \times \left(1 - \frac{\delta^2}{2N} \mathrm{Ric}(v)\right)$$

ON AVERAGE

RICCI CURVATURE, 2



$$\mathrm{Vol}(F_t) = \mathrm{Vol}(F_0)\left(1 - \frac{t^2}{2}\mathrm{Ric}(v) + O(t^3)\right)$$

RICCI CURVATURE
= CONTRACTION OF VOLUMES
BY THE GEODESIC FLOW

POSITIVE RICCI CURVATURE

LET M BE AN N-DIM RIEMANNIAN MANIFOLD WITH

$$\mathrm{Ric}(M) \geq K \, \mathrm{Ric}(S^N)$$

$(K>0)$

THEN:

- $\mathrm{DIAM}(M) \leq \dfrac{\pi}{\sqrt{K}}$

  (BONNET-MYERS)

- $\lambda_1(\Delta) \geq K\,N$

  (LICHNEROWICZ)

Δ = LAPLACE-BELTRAMI OPERATOR ON M

- $\forall f : M \to \mathbb{R}$ 1-LIPSCHITZ

$\exists m \in \mathbb{R}$

$$\mu\left(x \in M,\ |f(x) - m| > tD\right) < 2 \exp -\frac{t^2}{2}$$

WITH $D \simeq \dfrac{1}{\sqrt{NK}}$

(LEVY-GROMOV)

(CONSEQUENCE OF ISOPERIMETRIC INEQUALITY)

POSITIVE RICCI CURVATURE 3

LET P_t, $t \geq 0$ BE THE HEAT SEMI GROUP ON M.

- $\sup |\nabla P_t f| \leq e^{-tK(N-1)} \sup |\nabla f|$
- $|\nabla P_t f|(x) \leq e^{-tK(N-1)} P_t |\nabla f|(x)$
- $\operatorname{Ent} f \leq \dfrac{1}{2K(N-1)} \displaystyle\int_M \frac{|\nabla f|^2}{f}$

(LSI)

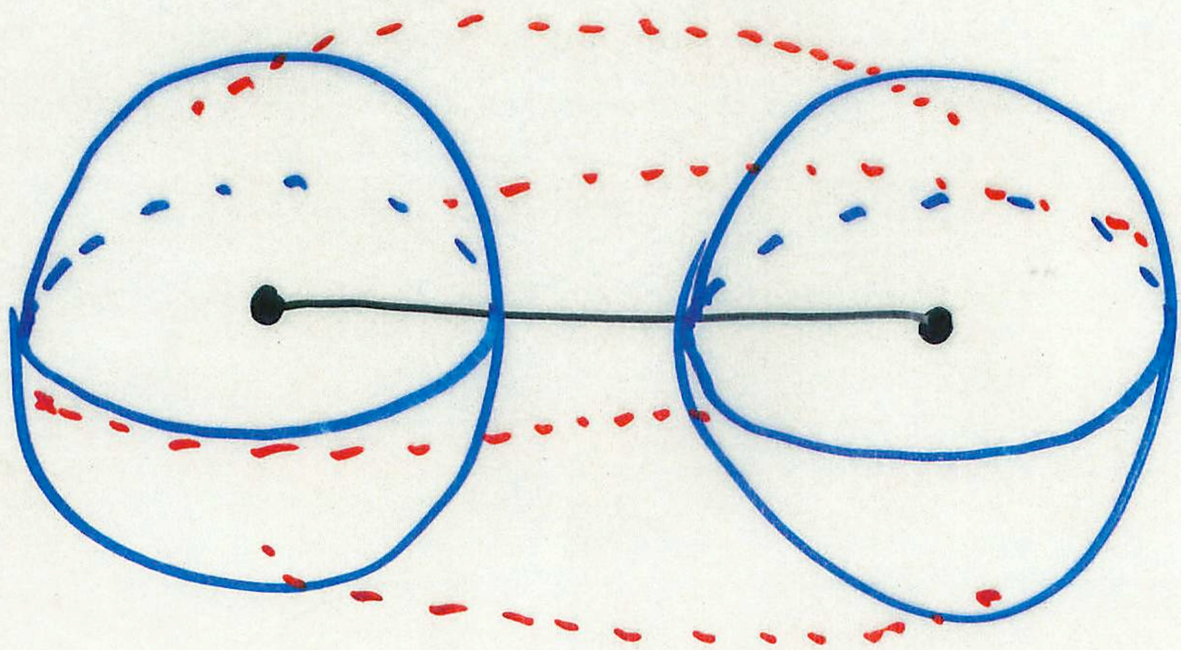
(BAKRY-EMERY)

WHERE $\operatorname{Ent} f := \displaystyle\int f \log \frac{f}{\int f}$

RICCI >0

$\Updownarrow$

BALLS ARE CLOSER THAN THEIR CENTERS ARE



DEF

(X, d) POLISH METRIC SPACE. A **SYSTEM OF BALLS** IS A FAMILY $(m_x)_{x \in X}$ WHERE EACH m_x IS A PROBABILITY MEASURE ON X.

$\rightsquigarrow$ **MARKOV CHAIN** ON X:

$$x_{t+1} \underset{\text{law}}{\sim} m_{x_t}$$

DEF

LET $x, y \in X$. THE **COARSE RICCI CURVATURE** IN DIRECTION $(x y)$ IS $\kappa(x,y)$ DEFINED BY

$$\boxed{\mathcal{T}(m_x, m_y) = d(x,y)\left(1 - \kappa(x,y)\right)}$$

WHERE $\mathcal{T}(m_x, m_y)$ = WASSERSTEIN TRANSPORTATION DISTANCE

$$\mathcal{T}(\mu, \nu) = \inf \int d(x, x')\, d\pi(x, x')$$

π COUPLING BETWEEN μ AND ν

SOME HISTORY

DOBRUSHIN 1970: USE OF "VASERSHTEIN" DISTANCE FOR CONVERGENCE OF MARKOV FIELDS. LATER:
DOBRUSHIN-SHLOSMAN 1984
DOBRUSHIN 1994, CHEN-WANG 1994
BUBLEY-DYER 1997,
DJELLOUT-GUILLIN-WU 2004

BAKRY-EMERY 1984: RICCI CURVATURE FOR DIFFUSIONS.
MANY WORKS INCLUDING
RENESSE-STURM 2005
STURM / LOTT-VILLANI / OHTA 2007

CONVERGENCE OF THESE TWO LINES

CF. ALSO JOULIN (2007), OLIVEIRA

OVERVIEW OF THEOREMS

POSITIVE κ IMPLIES

- DIAMETER BOUNDS (BONNET-MYERS)
- SPECTRAL GAP BOUND (LICHNEROWICZ)
- CONCENTRATION OF MEASURE (LEVY-GROMOV)
- GRADIENT NORM CONTRACTION
- MODIFIED LSI (BAKRY-EMERY)
- CONVERGENCE OF EMPIRICAL MEAN / MCMC (NEW?) (JOULIN + YO)

WITH **ESSENTIALLY SHARP** CONSTANTS.

\+ GROMOV-HAUSDORFF CV

\+ NON-NEGATIVE $\kappa \Rightarrow$ EXPONENTIAL CONCENTRATION (WITH SOME ASSUMPTIONS)

CHOICE OF m_x

EXAMPLE

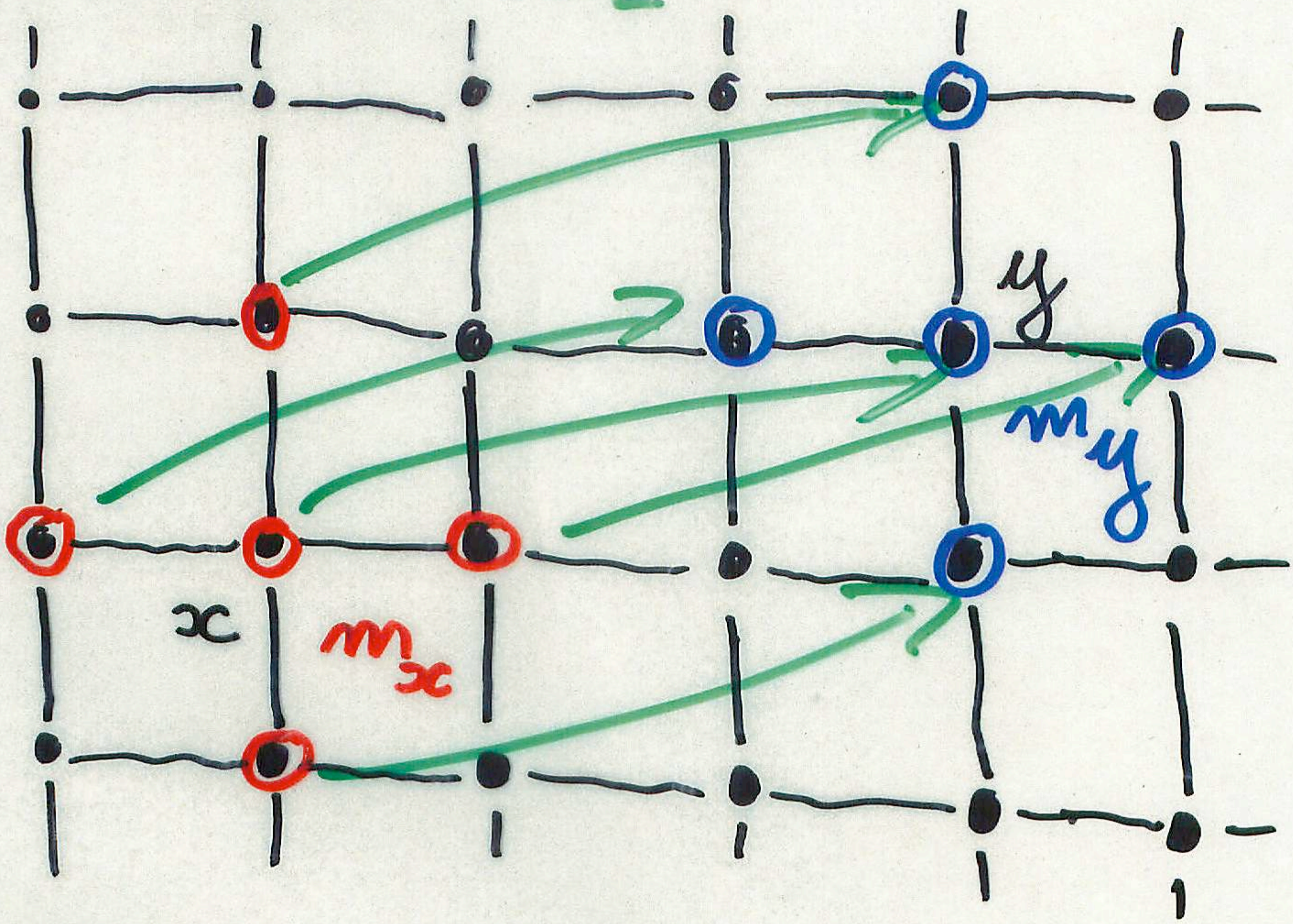
(X, d, μ) METRIC MÆASURE SPACE. CHOOSE $a > 0$.
TAKE

$$m_x := \frac{\mu|B(x,a)}{\mu(B(x,a))}$$

$\rightsquigarrow$ RICCI CURVATURE "AT SCALE a"

MANIFOLD : $a \to 0$
GRAPH : $a = 1$

LATTICE



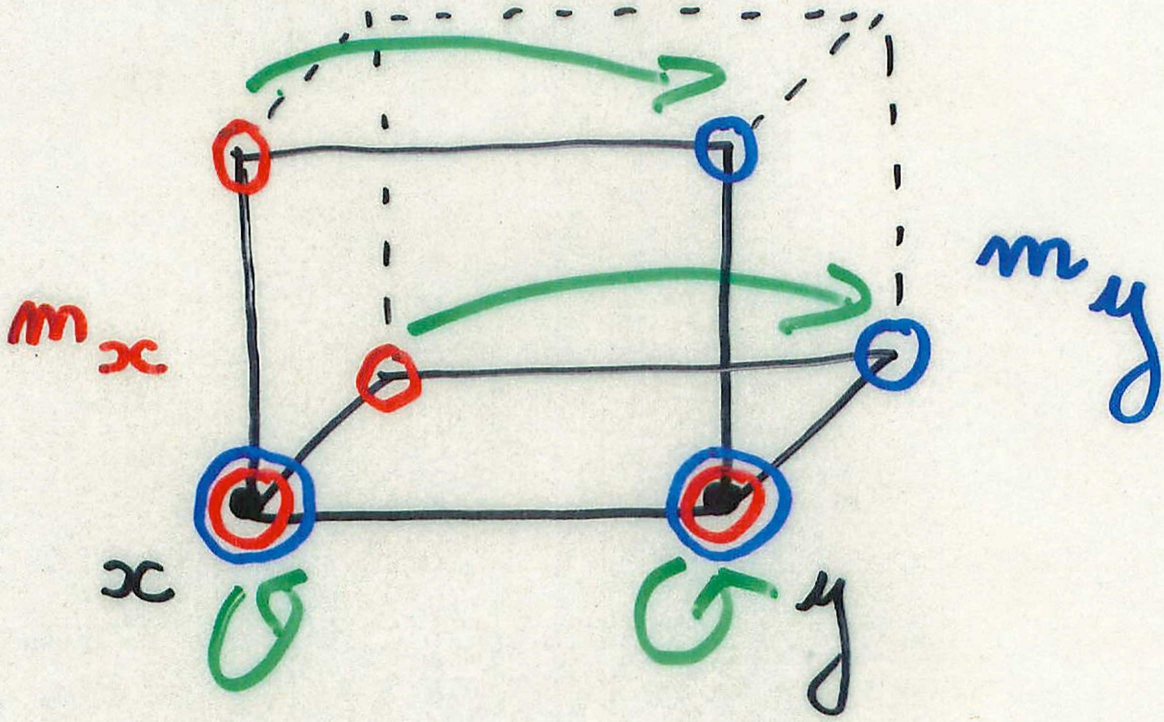
$$\tau(m_x, m_y) = d(x, y)$$

$$\Rightarrow \boxed{K = 0}$$

BY TRANSLATION.

SAME FOR $\mathbb{R}^N$ WITH ANY NORM

DISCRETE CUBE



$$\tau(m_x, m_y) = \frac{N-1}{N+1} \times 1 + \frac{2}{N+1} \times 0$$

$$= 1 - \frac{2}{N+1}$$

$$\kappa = \frac{2}{N+1}$$

HOW TO CHECK $\kappa > 0$?

EXERCISE

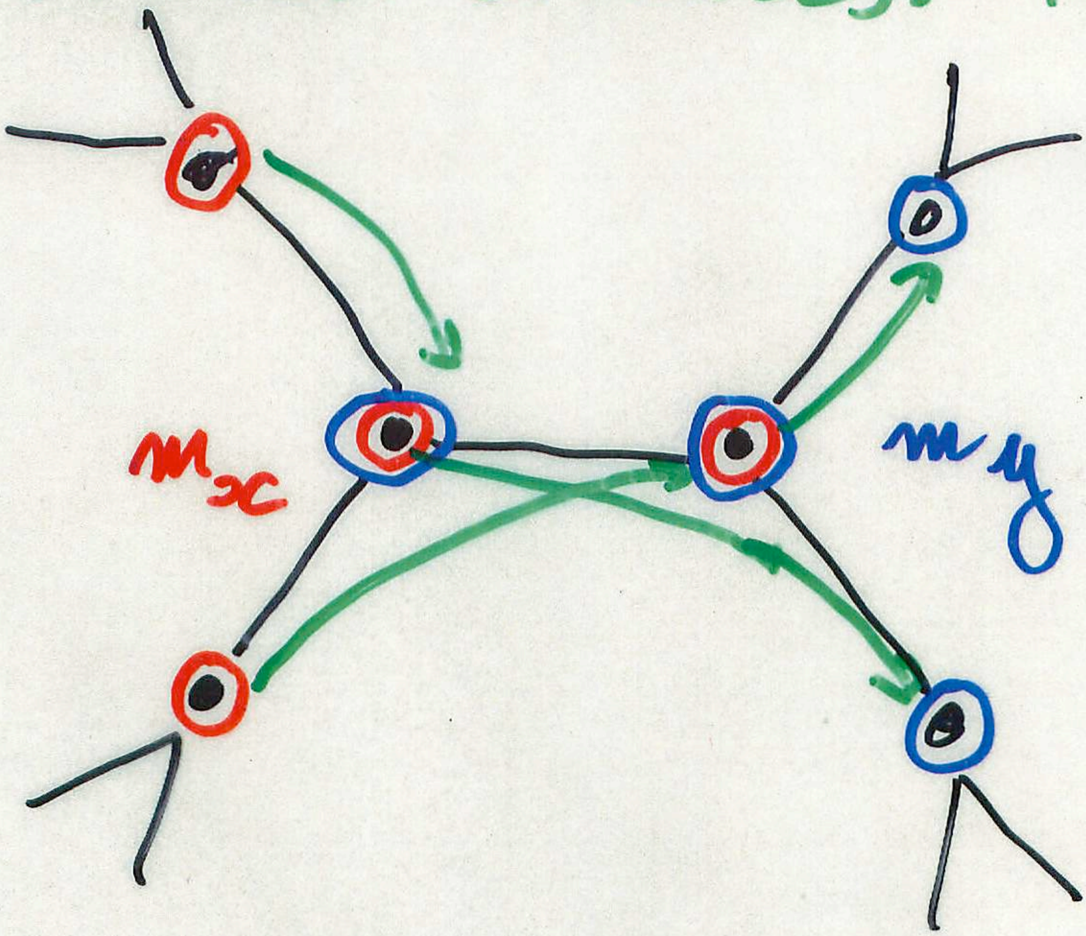
IF (X, d) IS ε-GEODESIC THEN $\kappa(x,y) > \kappa_0$ FOR $d(x,y) \leq \varepsilon$ IMPLIES $\kappa(x,y) > \kappa_0$ FOR ALL x, y.

ε-GEODESIC: FOR ALL x, x' $\exists$ $x = x_0, x_1, \dots x_k = x'$ WITH $d(x_i, x_{i+1}) \leq \varepsilon$ AND $d(x, x') = \sum d(x_i, x_{i+1})$

GRAPH : 1-GEODESIC

MANIFOLD : ε-GEODESIC $\forall \varepsilon > 0$

FURTHER EXAMPLES: TREES



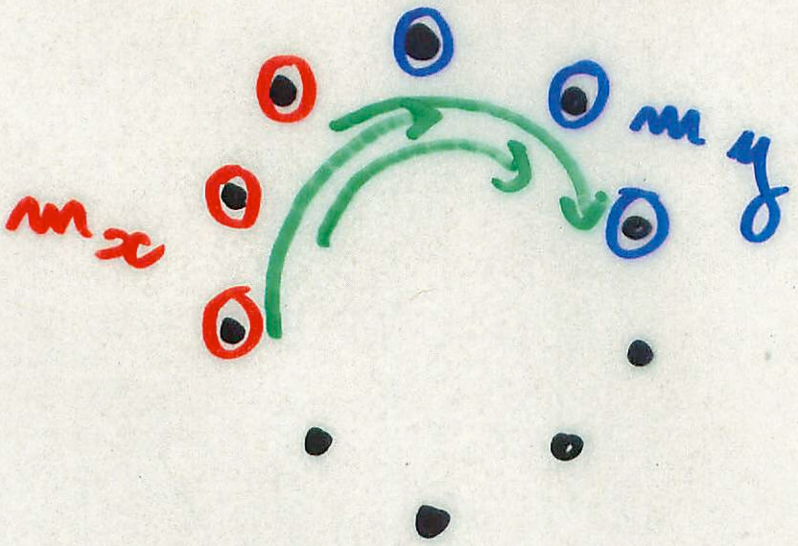
$$\tau(m_x, m_y) > d(x, y)$$

$$\Longrightarrow \boxed{\kappa < 0}$$

SAME FOR δ-HYPERBOLIC SPACES

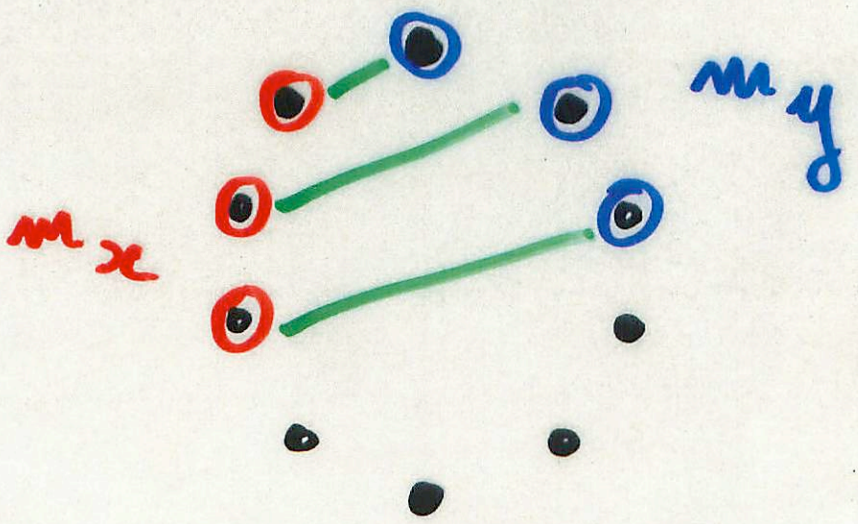
INFLUENCE OF METRIC

LENGTH-N CYCLE WITH GRAPH METRIC



$K = 0$

LENGH-N CYCLE WITH INDUCED $\mathbb{R}^2$ METRIC

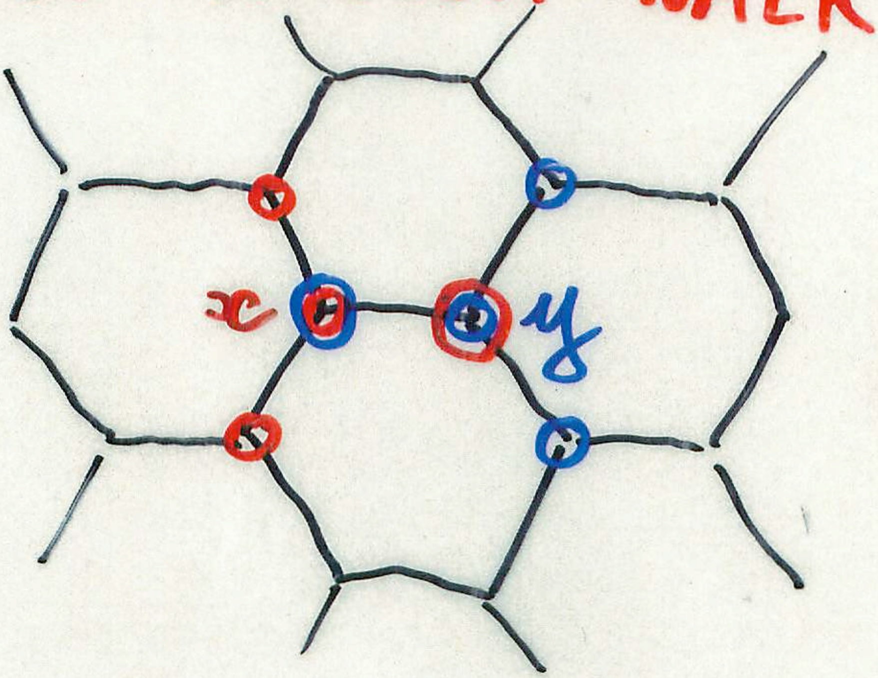


$K > 0$

⇝ EXTRINSIC CURVATURE

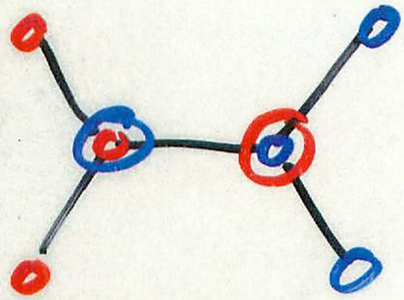
INFLUENCE OF m_x

SIMPLE RANDOM WALK

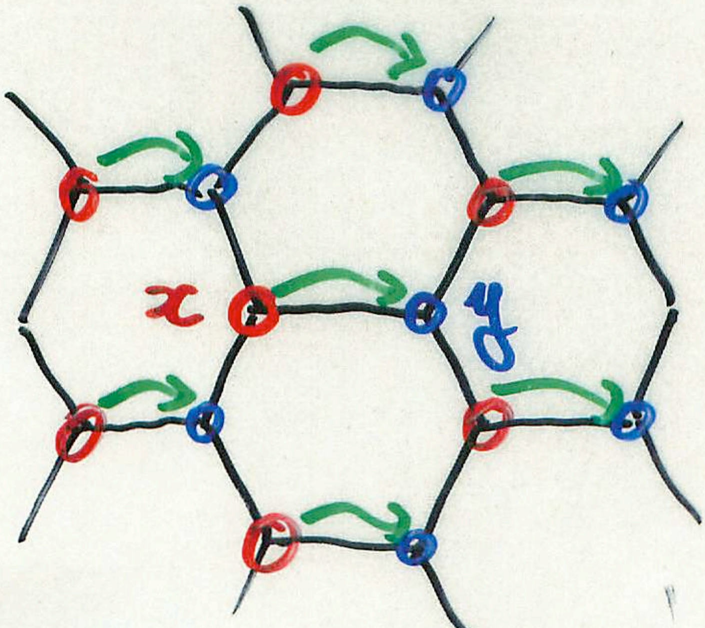


$\boxed{K<0}$

~TREE



TWO-STEP RANDOM WALK



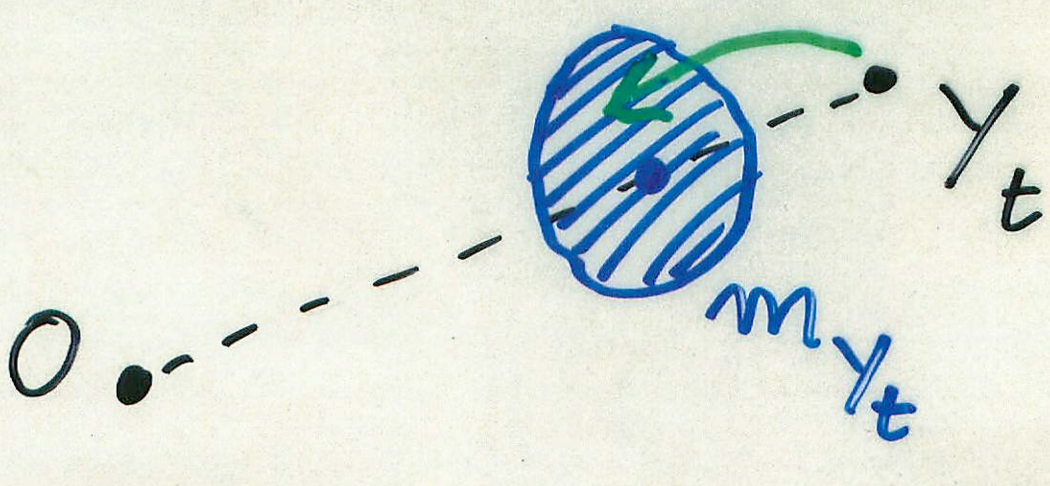
$\boxed{K=0}$

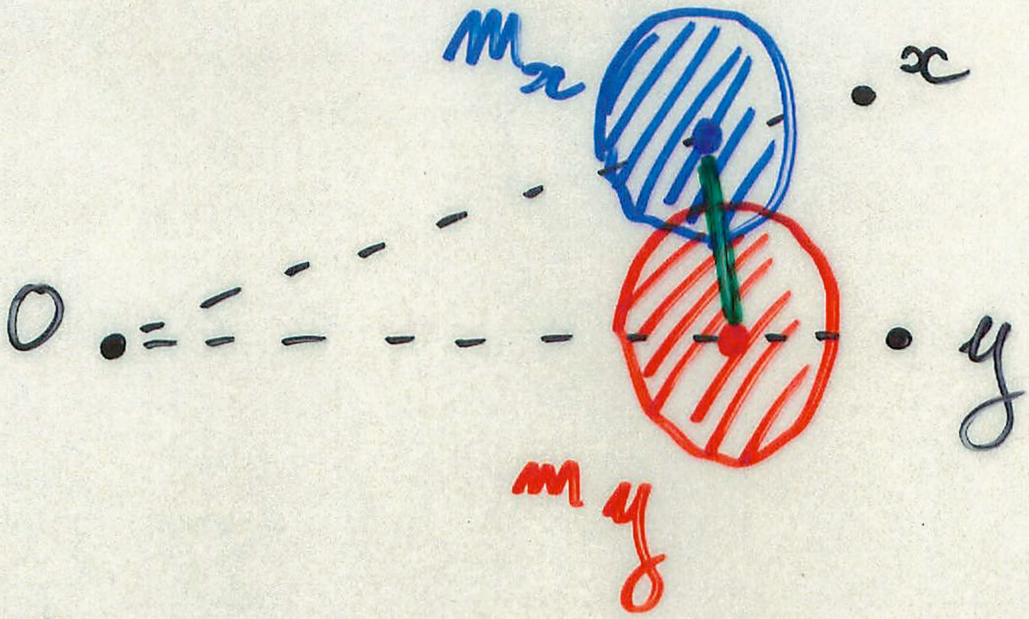
INFLUENCE OF MEASURE

GAUSSIAN MEASURE ON $\mathbb{R}^N$
= INVARIANT MEASURE OF THE ORNSTEIN-UHLENBECK PROCESS

$$dY_t = -Y_t\,dt + dB_t$$

FOR m_x : TAKE A RANDOM WALK WHICH APPROXIMATES THIS SDE (AS IN A NUMERICAL SIMULATION)





$$\mathcal{T}(m_x, m_y) < d(x,y)$$

$$\Rightarrow \quad \boxed{\kappa > 0}$$

SDE's ON MANIFOLDS

CONSIDER THE SDE

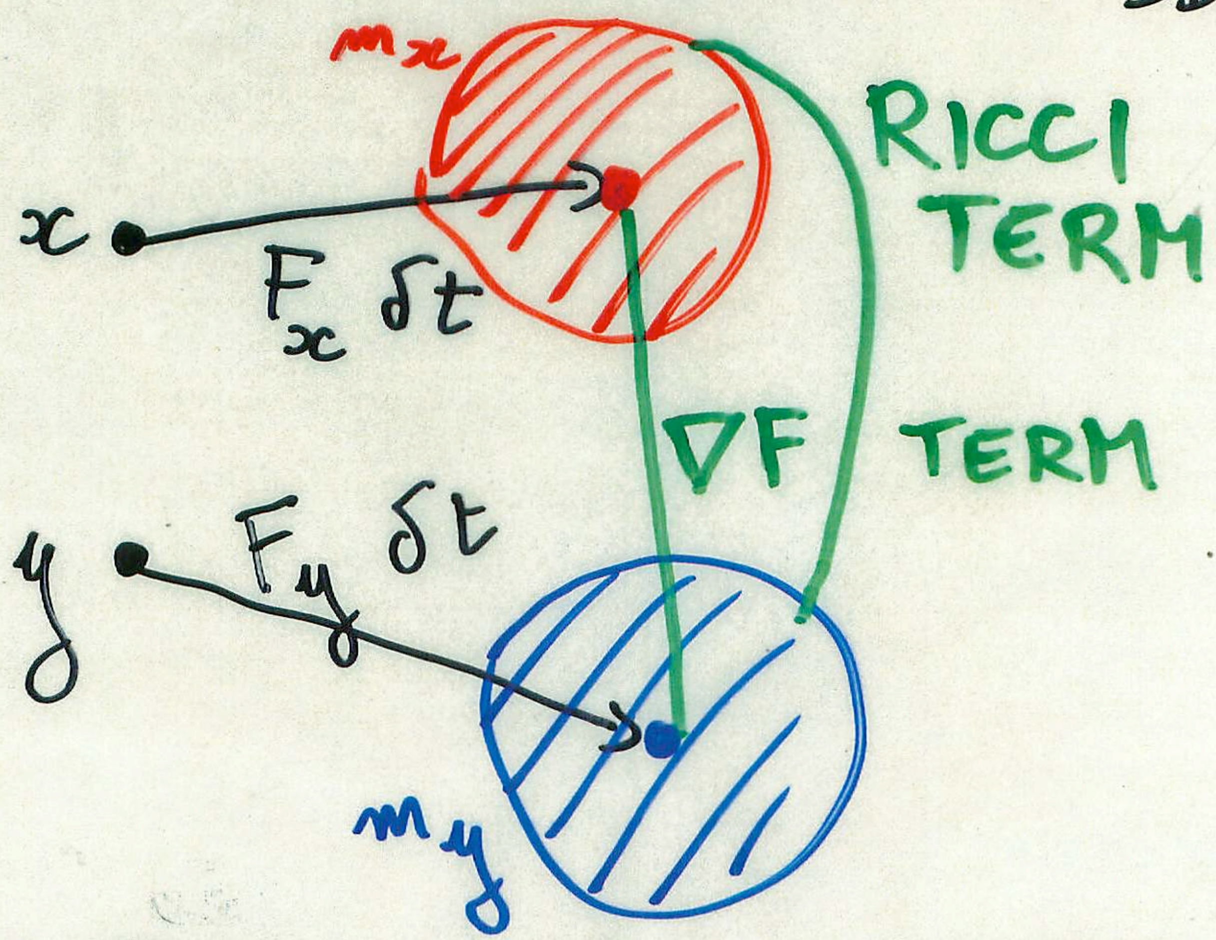
$$dX_t = F\,dt + dB_t$$

WITH GENERATOR

$$L = F.\nabla + \frac{1}{2}\Delta$$

ON A MANIFOLD.

m_x : DISCRETE APPROX. OF SDE



$\rightsquigarrow$ BAKRY-EMERY TENSOR

FURTHER EXAMPLE: ISING MODEL

G FINITE GRAPH

CONFIGURATION SPACE $X = \{+,-\}^G$

ENERGY $E(\sigma) = -\beta \sum_{g \sim g'} \sigma_g \sigma_{g'}$

GIBBS MEASURE $\mu(\sigma) \propto e^{-E(\sigma)}$

MARKOV CHAIN WITH μ AS INVARIANT MEASURE: THERMAL BATH

$$\sigma \longrightarrow \begin{cases} \sigma_{x+} & \text{WITH PROBA } \dfrac{\mu(\sigma_{x+})}{\mu(\sigma_{x+})+\mu(\sigma_{x-})} \\ \sigma_{x-} & \text{WITH PROBA } \dfrac{\mu(\sigma_{x-})}{\mu(\sigma_{x+})+\mu(\sigma_{x-})} \end{cases}$$

FOR $x \in G$ CHOSEN AT RANDOM.

$\rightsquigarrow m_\sigma$

PROP

$$K > \frac{1}{\# G}\left(1 - \deg_{max}\ th(2\beta)\right)$$

IN PARTICULAR $K>0$ FOR $\beta < \ldots$

$\rightsquigarrow$ EASILY RECOVER KNOWN RESULTS ABOUT ISING MODEL AND ITS VARIANTS.

MARKOV CHAINS

$m_x \rightsquigarrow$ MARKOV CHAIN WITH TRANSITION KERNEL AT TIME t

$$m_x^{*t}(dy) = \int m_x^{*t-1}(dz)\, m_z(dy)$$

WITH $m_x^{*1} := m_x$

NOTATION: μ MEASURE ON X

$$\mu * m\,(dy) := \int \mu(dz)\, m_z(dy)$$

"START WITH μ AND MAKE ONE STEP"

PROP:

IF $\kappa(x, y) \geq \kappa \quad \forall x, y$ THEN

$$\mathcal{T}(\mu_1 * m, \mu_2 * m) \leq (1-\kappa)\, \mathcal{T}(\mu_1, \mu_2)$$

COR: $\exists!$ INVARIANT MEASURE ν.

COR: $\mathcal{T}(m_x^{*t}, \nu) \leq (1-\kappa)^t \times \mathrm{Diam}\, X$

$\rightsquigarrow$ MIXING TIME ESTIMATES

FOR EX. $N \log(N)$ FOR THE CUBE

DEF: AVERAGING OPERATOR

$$(Mf)(x) := \int f(y) \; m_x(dy)$$

PROP: $\kappa(x,y) \geq \kappa \quad \forall (x,y)$

IFF

$$\|Mf\|_{Lip} \leq (1-\kappa) \|f\|_{Lip}$$

COR: LET $\Delta := M - Id$.

ASSUME THE RANDOM WALK DEFINED BY (m_x) IS REVERSIBLE

THEN

$$\lambda_1(\Delta) \geq \kappa$$

(cf. LICHNEROWICZ)

CONCENTRATION THEOREM

THM (007)

SUPPOSE $\kappa(x,y) \geqslant \kappa > 0$.

THEN $\forall f : X \to \mathbb{R}$ 1-LIPSCHITZ

$$\nu\left(x, |f(x) - \mathbb{E}_\nu f| > t\right) \leq \exp - \frac{t^2}{2D^2}$$

FOR $0 \leq t \leq t_{max}$.

ν: INVARIANT MEASURE

t_{max}: EXPLICIT, DEPENDS ON m_x
(USUALLY LARGE)

$$D^2 = \mathbb{E}_\nu D_x^2$$

WITH $D_x = \dfrac{\sigma_x}{\sqrt{n_x \kappa}}$

SIZE OF m_x

LOCAL DIMENSION

$$\sigma_x = \sqrt{\frac{1}{2} \iint d(y,y')^2 \, m_x(dy)\, m_x(dy')}$$

$\rightsquigarrow$ AVERAGE SQUARE DISTANCE BETWEEN POINTS OF m_x

EXAMPLES

- X = MANIFOLD, m_x = ε-BALL
  $\Rightarrow \sigma_x \asymp \varepsilon$
- X = GRAPH, m_x = SIMPLE RANDOM WALK
  $\Rightarrow \sigma_x \asymp 1$

COARSE DIMENSION

(X, d, μ) METRIC MEASURE SPACE.

DEFINE

$$\mathrm{COARSEDIM}(X, d, \mu) = \frac{\iint d(y,y')^2 \, \mu(dy)\, \mu(dy')}{\sup \iint \big(f(y) - f(y')\big)^2 \mu(dy)\, \mu(dy')}$$

$f : X \to \mathbb{R}$ 1-LIPSCHITZ

LOCAL DIMENSION

$$n_x = \mathrm{COARSEDIM}(X, d, m_x)$$

EXAMPLES

- N-DIM MANIFOLD: $n_x \asymp N$
- GRAPH : $n_x \asymp 1$

BACK TO CONCENTRATION

- N-DIM MANIFOLD WITH $\mathrm{Ric} \geqslant K\, \mathrm{Ric}(\text{SPHERE})$

$$\Rightarrow \left.\begin{aligned} \kappa &\asymp K\varepsilon^2 \\ \sigma_x &\asymp \varepsilon \\ n_x &\asymp N \end{aligned}\right\} D \asymp \frac{\sigma_x}{\sqrt{n_x \kappa}} \asymp \frac{1}{\sqrt{NK}}$$

AND $t_{max} \to \infty$

- N-DIM CUBE (WITH DIAM = 1)

$$\Rightarrow \left.\begin{aligned} \kappa &\asymp 1/N \\ \sigma_x &\asymp 1/N \\ n_x &\asymp 1 \end{aligned}\right\} \frac{\sigma_x}{\sqrt{n_x \kappa}} \asymp \frac{1}{\sqrt{N}}$$

AND $t_{max} > 1$

- BINOMIAL / POISSON: OK, CORRECT VARIANCE AND t_{max}

- $t \geqslant t_{max}$: EXPONENTIAL CONCENTRATION

LOG-SOB INEQUALITY

NEED A COARSE GRADIENT.

CHOOSE $\lambda > 0$. ("INVERSE SCALE")

DEFINE

$$\nabla_\lambda f(x) = \sup_{y,y'} \frac{|f(y) - f(y')|}{d(y,y')} \cdot C(x,y,y')$$

WITH

$$C(x,y,y') = e^{-\lambda d(x,y) - \lambda d(y,y')}$$

- $\lambda = 0$: LIPSCHITZ NORM
- $\lambda \to \infty$: NORM OF GRADIENT

DESIGNED SO THAT THE HERBST ARGUMENT FOR LSI $\Rightarrow$ CONCENTRATION STOPS AT λ.

GRADIENT CONTRACTION + LSI

THM (007)

SUPPOSE $\kappa(x,y) \geqslant \kappa > 0$.
TAKE $\lambda < \lambda_{max}$.
THEN $\forall f : X \to \mathbb{R}$

$$\nabla_\lambda Mf(x) \leq \left(1 - \frac{\kappa}{2}\right) M \nabla_\lambda f(x)$$

WHERE M = AVERAGING OPERATOR

MOREOVER FOR $f > 0$

$$\mathrm{Ent}\, f \leqslant 4\left(\sup \frac{\sigma_x^2}{n_x \kappa}\right) \int \frac{(\nabla_\lambda f)^2}{f}$$

W.R.T. INVARIANT MEASURE

- MANIFOLD: $\lambda_{max} \to \infty$
- CUBE: $\lambda_{max} \asymp \dfrac{1}{\text{EDGE LENGTH}}$

MCMC/CONVERGENCE OF EMPIRICAL MEANS

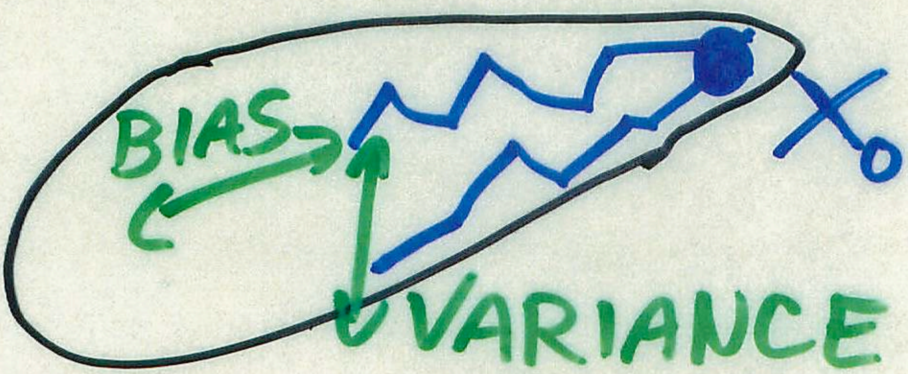
GOAL: ESTIMATE $\mathbb{E}_\nu f$ WITHOUT SAMPLING FROM ν.

EMPIRICAL MEAN

$$\hat{f}_T(x_0) = \frac{1}{T}\sum_{i=0}^{T-1} f(x_i)$$

WHERE $x_{i+1} \sim m_{x_i}$

ERROR $\rightsquigarrow$ BIAS + VARIANCE



TO REDUCE BIAS, REPLACE WIT

$$\hat{f}_{T_0,T} = \frac{1}{T}\sum_{i=T_0}^{T_0+T-1} f(x_i)$$

BIAS ESTIMATION

PROP ASSUME $\kappa > 0$. THEN

$$\left|\mathbb{E}_{x_0} \hat{f}_{T_0,T} - \mathbb{E}_\nu f\right| \leq (1-\kappa)^{T_0} \frac{E(x_0)}{T\kappa}$$

$\boxed{f \text{ 1-LIP}}$

WHERE E IS THE ECCENTRICITY

$$E(x_0) := \mathcal{T}(\delta_{x_0}, \nu)$$

- $E(x_0) \leq$ DIAM
- $E(x_0) \leq E(\theta) + d(\theta, x_0)$
- $E(x_0) \leq \frac{1}{\kappa} \int d(x_0, x_1)\, m_{x_0}(dx_1)$

VARIANCE

THM (JOULIN - O 08). SUPPOSE $\kappa > 0$.

$$\mathrm{Var}\ \hat{f}_{T_0,T} \leq \frac{8}{\kappa T} \sup_x \frac{\sigma_x^2}{n_x \kappa}$$

+ GAUSSIAN / EXPONENTIAL CONCENTRATION

FOR ANY 1-LIPSCHITZ FUNCTION f.

REMEMBER, $\dfrac{\sigma_x^2}{n_x \kappa} \rightsquigarrow$ VARIANCE OF THE INVARIANT DISTRIBUTION.

COROLLARY

LET X_t BE THE BROWNIAN MOTION ON A RIEMANNIAN MANIFOLD WITH $\mathrm{Ric} \geq K$. LET f BE A 1-LIPSCHITZ FUNCTION AND LET

$$\hat{f}_T(x_0) := \frac{1}{T} \int_{t=0}^{T} f(x_t)\, dt$$

THEN

$$\Pr\left(\hat{f}_T(x_0) - \mathbb{E}\hat{f}_T(x_0) \geq r \right) \leq \exp - \frac{K^2 T r^2}{32}$$

CF. GUILLIN-LEONARD-WU-YAO

OPEN PROBLEMS

- LINK WITH STURM-LOTT-VILLANI?
- ALEXANDROV SPACES $\Rightarrow \kappa \geqslant 0$?
- NILPOTENT GROUPS ?
- SPECTRAL GAP IN NON-REVERSIBLE CASE ?
- THIS IS A DISCRETE $CD(\kappa, \infty)$.
  DEFINE A DISCRETE $CD(\kappa, N)$.
- DISCRETE RICCI FLOW?
- DISCRETE SECTIONAL / SCALAR CURVATURE?
- EXPANDERS $\Rightarrow \kappa < 0$?
- EXPLORE THE FINSLER CASE.
- ...